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Effect of Landau level mixing on the effective interaction between electrons in the fractional quantum Hall regime

Waheb Bishara[1] and Chetan Nayak[2,3]

[1]*Department of Physics, California Institute of Technology, MC 256-80, Pasadena, California 91125, USA*

[2]*Microsoft Research, Station Q, CNSI Building, University of California, Santa Barbara, California 93106-4030, USA*

[3]*Department of Physics, University of California, Santa Barbara, California 93106, USA*

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We compute the effect of Landau-level mixing on the effective two-body and three-body pseudopotentials for electrons in the lowest and second Landau levels. We find that the resulting effective three-body interaction is attractive in the lowest relative angular momentum channel. The renormalization of the two-body pseudopotentials also shows interesting structure. We comment on the implications for the $\nu=5/2$ fractional quantum Hall state.

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Deep in the quantum Hall regime when $\hbar\omega_c=\hbar eB/mc$ is the largest energy scale, the Hamiltonian of a two-dimensional electron liquid can be projected into a single Landau level (LL), with all other LLs being either completely filled or completely empty. The projected Hamiltonian is often tractable by numerical diagonalization[1–5] or, more recently, the density-matrix renormalization group (DMRG).[6,7] Since the early days of the fractional quantum Hall effect, such numerical studies have played a major role in establishing the viability of various theories of observed quantum Hall plateaus.

The neglect of completely filled and empty LLs is a good approximation so long as the Coulomb interaction scale $e^2/4\pi\epsilon\ell_0=(e^2/4\pi\epsilon)\sqrt{eB/\hbar c}$ is much smaller than $\hbar\omega_c \equiv \hbar eB/mc$. Corrections due to Landau-level mixing can be computed in powers of $\kappa\equiv(e^2/4\pi\epsilon\ell_0)/\hbar\omega_c\propto 1/\sqrt{B}$.[8–10] Experiments in GaAs heterostructures and quantum wells are done at magnetic fields $\sim 2-15$ T, where fractional quantum Hall states such as $\nu=1/3$ and $\nu=5/2$ are typically observed, which implies that $\kappa\sim 0.6-1.8$. Thus, one might expect that the effects of Landau-level mixing would not be small. Furthermore, even if small, Landau-level mixing (along with disorder) is the leading effect which explicitly breaks particle-hole symmetry within a LL. This may be of particular importance in the second LL. The $\nu=5/2$ state[11–14] may be *inter alia* one of two candidates connected by particle-hole conjugation, the Moore-Read (MR) Pfaffian[15–17] state and the anti-Pfaffian state.[18,19] In addition, a plateau is observed at $\nu=12/5$ but not at the particle-hole conjugated fraction, $\nu=13/5$.[14] Numerical studies of both 5/2 (Refs. 7 and 20–22) and 12/5 have, thus far, neglected the effects of LL mixing and may, thus, be questioned.

Thus, in this Rapid Communication, we compute the renormalization of the effective interaction within a LL due to virtual excitations of electrons into higher LLs and holes into lower ones. We focus on the effective two- and three-body interactions which are generated in this way at lowest order in κ. However, our analysis can be generalized in a straightforward way to four-body and higher interactions which are generated at higher order in κ. We present our results in terms of Haldane's two-body pseudopotentials[4] and the generalization to three-body and higher pseudopotentials due to Simon *et al.*,[23] which are the projections of the interaction onto states of fixed relative angular momentum. Our results can be used as an input for future numerical studies of states in both the lowest and second LLs. Our results have several interesting features. We find that the effective interactions due to LL mixing are, at lowest order, $\propto\kappa\cdot(e^2/4\pi\epsilon\ell_0)$ with a coefficient which is small. Thus, even though the expansion parameter κ is O(1), a perturbative expansion may be valid. In the lowest angular momentum channel relevant to spin-polarized electrons, the LL mixing contribution to the three-body pseudopotentials is negative, i.e., is attractive.

We begin with the action

$$S=\int\frac{d\omega}{2\pi}\sum_{m,n}\bar{c}^{\alpha}_{m,n}(\omega)(i\omega-E_n+\mu)c^{\alpha}_{m,n}(\omega)$$
$$-\frac{1}{2}\int\prod_{i=0}^{4}\frac{d\omega_i}{2\pi}\frac{e^2}{4\pi\epsilon\ell_0}V_{4,3;2,1}\bar{c}^{\alpha}_{m_4,n_4}(\omega_4)\bar{c}^{\beta}_{m_3,n_3}(\omega_3)$$
$$\times c^{\beta}_{m_2,n_2}(\omega_2)c^{\alpha}_{m_1,n_1}(\omega_1)\delta^{\omega}_{43,21}, \tag{1}$$

where $c_{m,n}$ and $\bar{c}_{m,n}$ are Grassman variables, the m_is are an orbital indices distinguishing states within a LL and the n_is are LL indices. The spin indices $\alpha,\beta=\uparrow,\downarrow$ are summed over whenever repeated. $\delta^{\omega}_{43,21}$ is shorthand for $2\pi\delta(\omega_4+\omega_3-\omega_2-\omega_1)$ and $V_{4,3;2,1}$ is shorthand for $V(m_4,n_4;m_3,n_3;m_2,n_2;m_1,n_1)$. The single-particle energies are $E_n=\hbar\omega_c(n+\frac{1}{2})$, and μ is the chemical potential, which we assume to be equal to $\hbar\omega_c(N+1/2)$. Changes in the chemical potential $\sim e^2/4\pi\epsilon\ell_0$, which change the fractional filling of the Nth LL, do not affect the renormalization of the effective interaction to lowest order in κ. We have set the Zeeman energy to zero since it is much smaller than either the Coulomb or cyclotron energies. Spontaneous spin polarization in the Nth LL is not precluded by this approximation, and in any case, it is straightforward to restore the Zeeman energy. The Coulomb interaction matrix elements are given by the following expression, where $G_{a,b}(q)$, for $a>b$, is[24] $G_{a,b}(q)=(b!/a!)(-iq/\sqrt{2})^{a-b}L_b^{a-b}(|q|^2/2)$, where $L_b^{a-b}(q)$ is the generalized Laguerre polynomial:

$$V_{4,3;2,1}=\int\frac{d^2q}{(2\pi)^2}\frac{2\pi}{|q|}e^{-q^2}G_{n4,n1}(q^*)G_{n3,n2}(-q^*)$$
$$\times[G_{m4,m1}(-q)]^2. \tag{2}$$

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We assume that the Nth LL (specializing later to $N=0,1$ LLL, and SLL) is partially filled and integrate out all higher and lower LLs to obtain the effective action S_{eff} in the Nth LL:

$$e^{iS_{\text{eff}}[\bar{c}^{\alpha}_{m,N},c^{\alpha}_{m,N}]}=\int\prod_{n\neq N}\prod_{m}[d\bar{c}_{m,n}(\omega)dc_{m,n}(\omega)]e^{iS}. \quad (3)$$

We carry out this integration perturbatively in the Coulomb interaction which, as we will see, amounts to an expansion in κ. To first-order in this expansion parameter, S_{eff} will take the form

$$\begin{aligned}S=&\int\frac{d\omega}{2\pi}\sum_{m}\bar{c}^{\alpha}_{m,N}(\omega)(i\omega-E_N+\mu)c^{\alpha}_{m,N}(\omega)\\&-\frac{1}{2}\int\prod_{i=0}^{4}\frac{d\omega_i}{2\pi}\frac{e^2}{4\pi\epsilon\ell_0}u_2(m_4,m_3,m_2,m_1)\delta^{\omega}_{43,21}\\&\times\bar{c}^{\alpha}_{m_4,N}(\omega_4)\bar{c}^{\beta}_{m_3,N}(\omega_3)c^{\beta}_{m_2,N}(\omega_2)c^{\alpha}_{m_1,N}(\omega_1)\\&-\frac{1}{3!}\int\prod_{i=0}^{6}\frac{d\omega_i}{2\pi}\frac{e^2}{4\pi\epsilon\ell_0}u_3(m_6,m_5,m_4,m_3,m_2,m_1)\\&\times\delta^{\omega}_{654,321}\bar{c}^{\alpha}_{m_6,N}(\omega_6)\bar{c}^{\beta}_{m_5,N}(\omega_5)\bar{c}^{\gamma}_{m_4,N}(\omega_4)\\&\times c^{\gamma}_{m_3,N}(\omega_3)c^{\beta}_{m_2,N}(\omega_2)c^{\alpha}_{m_1,N}(\omega_1).\end{aligned} \quad (4)$$

At zeroth order, $u_2(m_4,m_3,m_2,m_1)$ is simply the bare Coulomb interaction projected into the Nth LL, $V(m_4,N;m_3,N;m_2,N;m_1,N)$. This is renormalized at one loop, as we discuss below. Meanwhile, $u_3(m_6,m_5,m_4,m_3,m_2,m_1)$ is not present at zeroth order and is only generated when we take into account virtual transitions into higher LLs. At higher orders, four-body, five-body, etc. interactions $u_4,u_5,\ldots$ will be generated, but we do not discuss these here.

Our perturbative calculation is facilitated by the use of Feynman diagrams, obeying the following rules. We will always be considering diagrams in which the Coulomb interaction enters in the combination $\tilde{V}^{\alpha'\beta';\beta\alpha}_{4,3;2,1}=V_{4,3;2,1}\delta^{\alpha\alpha'}\delta^{\beta\beta'}-V_{3,4;2,1}\delta^{\alpha\beta'}\delta^{\beta\alpha'}$ shown in Fig. 1(a). If we define a single vertex as the difference between the two vertices shown in Fig. 1(a), then there is only a single vertex in our calculations, with two electron lines going in and two coming out. It receives the vertex factor $\tilde{V}^{\alpha'\beta';\beta\alpha}_{4,3;2,1}\delta^{\omega}_{43,21}$, where m_1 and m_2 are the orbital indices of the incoming electrons and m_3 and m_4 are the orbital indices of the outgoing electrons. In the diagrams of interest to us, all but one or two of the particles' LL indices n_i will be that of the partially filled LL, N, while the internal legs will have LL indices which range freely over $n\neq N$. Every internal electron line receives the factor $1/(i\omega-E_n)$, where ω is the frequency of the line and $E_n=(n+\frac{1}{2})\hbar\omega_c$. The lowest order diagrams with four and six external lines, respectively, are in Figs. 1(b)–1(d). To compute the effective interaction in the Nth LL, we require that all external legs in these diagrams have $n_{\text{ext}}=N$. The internal legs range over all $n\neq N$. For $N=1$, this includes both virtual transitions of holes to the LLL and of electrons to $N\geq 2$ LLs.

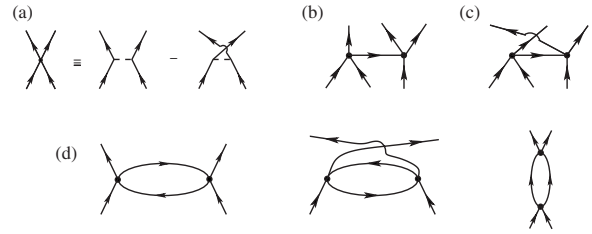

FIG. 1. (a) The solid directed lines are electron lines and the dotted line denotes the Coulomb interaction. We define the four-point vertex $\tilde{V}^{\alpha'\beta';\beta\alpha}_{4,3;2,1}=V_{4,3;2,1}\delta^{\alpha\alpha'}\delta^{\beta\beta'}-V_{3,4;2,1}\delta^{\alpha\beta'}\delta^{\beta\alpha'}$ on the left-hand side because the combination on the right-hand side enters all of the diagrams which renormalize u_2 and u_3. (b) and (c) Two of the nine diagrams which renormalize u_3. (d) The three diagrams which renormalize u_2.

The diagrammatic expansion of this problem is similar to that of a Fermi liquid, with the exception that momentum integrals are replaced by sums over orbitals and Landau-level indices. This has an important effect, namely, that phase-space restrictions are much less severe. To see this, consider the lowest-order diagram, shown in Fig. 1(b) contributing to the three-body effective interaction. It corresponds to the following expression:

$$\left(\frac{e^2}{4\pi\epsilon\ell_0}\right)^2\sum_{m_x;n_x\neq N}\int\frac{d\omega_x}{2\pi}\frac{\tilde{V}^{\alpha'\lambda;\beta\alpha}_{6,x;2,1}\tilde{V}^{\beta'\gamma';\gamma\lambda}_{5,4;x,3}}{i\omega_x-(E_{n_x}-\mu)}\delta^{\omega}_{6x,32}\delta^{\omega}_{54,x1}. \quad (5)$$

The energy $E_{n_x}-\mu$ is approximately $\hbar\omega_c(n_x-N)$. As a result of the δ functions, the ω_x integral in this particular diagram simply enforces energy conservation. Note that there is a sum over LLs and orbitals even though this is a tree-level diagram. In a zero-field Fermi liquid, by contrast, there would be no momentum integral in such a diagram since the internal momentum would be completely fixed by the external ones. From the energy denominator, we see that the contribution from virtual holes in lower LLs is, naively, positive while the contribution from virtual electrons in higher LLs is, naively, negative. However, the matrix elements are not strictly positive so the situation is more complicated, as we will see in Table I.

In fact, there are nine diagrams similar to Fig. 1(b) which can be obtained from this one by permuting the external legs. Another one of them is shown in Fig. 1(c). Note that these nine diagrams are actually 36 diagrams when each vertex is expanded as in Fig. 1(a). These can be divided into six sets of six diagrams, one set for each possible spin index structure $\delta^{\pi(\alpha)\alpha'}\delta^{\pi(\beta)\beta'}\delta^{\pi(\gamma)\gamma'}$, where π is one of the six permutations of α, β, and γ. Summing them all up antisymmetrizes $u_3(m_6,\ldots,m_1)\delta^{\alpha\alpha'}\delta^{\beta\beta'}\delta^{\gamma\gamma'}$ under all permutations of $\{(m_1,\alpha);(m_2,\beta);(m_3,\gamma)\}$ and of $\{(m_4,\gamma');(m_5,\beta');(m_6,\alpha')\}$.

Performing the integral in Eq. (5), we find that the resulting three-body coupling in the effective Lagrangian is given by $-u_3(m_6,\ldots,m_1)\delta^{\omega}_{654,321}c^{\dagger}_6c^{\dagger}_5c^{\dagger}_4c_3c_2c_1$ [there is a minus sign since the contribution from Eq. (5) has been re-exponentiated and combined with $-S$] with u_3 given by

TABLE I. Lowest order three-particle pseudopotentials, $V^{(3)}_{m,s}$, and two-particle pseudopotentials, $\delta V^{(2)}_{m,s}$, for $N=0$, 1 (LLL and SLL). The pseudopotentials are in units of $\kappa e^2/4\pi\epsilon\ell_0$. The table on the top left contains the pseudopotentials for three particles with total spin $S=3/2$ (relevant, for instance, to the case of fully spin-polarized electrons) while the table on the top right is for three particles with $S=1/2$. The table on the bottom left contains the pseudopotentials for pairs of particles with total spin $S=1$ (relevant, for instance, to the case of fully spin-polarized electrons) while the table on the bottom right is for $S=0$.

m	$V^{(3)}_{m,3/2}$ (LLL)	$V^{(3)}_{m,3/2}$ (SLL)
3	−0.0181	−0.0147
5	0.0033	−0.0054
6	−0.0107	−0.0099
7	0.0059	0.0005
8	−0.0048	−0.0009

m	$V^{(3)}_{m,1/2}$ (LLL)	$V^{(3)}_{m,1/2}$ (SLL)
1	−0.0346	−0.0324
2	−0.0541	−0.0315

m	$\delta V^{(2)}_{m,1}$ (LLL)	$\delta V^{(2)}_{m,1}$ (SLL)
1	−0.0053	0.0042
3	−0.0004	0.0023

m	$\delta V^{(2)}_{m,0}$ (LLL)	$\delta V^{(2)}_{m,0}$ (SLL)
0	−0.1032	−0.0325
2	−0.0012	−0.0174
4	−0.0002	−0.0034

$$\sum_{\pi\in S_3}(-1)^{\pi}u_3(\pi(m_6),\pi(m_5),\pi(m_4),m_3,m_2,m_1)\times\delta^{\alpha\pi(\alpha')}\delta^{\beta\pi(\beta')}\delta^{\gamma\pi(\gamma')}=-\kappa\sum_{\omega,\theta'\in C_3}\left\{\sum_{m_x;n_x\neq N}\frac{\tilde{V}^{\theta(\alpha')\lambda;\omega(\beta)\omega(\alpha)}_{\theta(6),x;\omega(2),\omega(1)}\tilde{V}^{\theta(\beta')\theta(\gamma');\omega(\gamma)\lambda}_{\theta(5),\theta'(4);x,\omega(3)}}{n_x-N}\right\}. \tag{6}$$

On the left-hand side, we sum over permutations π of $\{(m_4,\gamma');(m_5,\beta');(m_6,\alpha')\}$; on the right-hand side, we sum over cyclic permutations of $\{(m_4,\gamma');(m_5,\beta');(m_6,\alpha')\}$ and of $\{(m_1,\alpha);(m_2,\beta);(m_3,\gamma)\}$ because the four-point vertex $\tilde{V}^{\alpha'\beta';\beta\alpha}_{4,3;2,1}$ has already been antisymmetrized. In this expression, we have approximated the energy denominator $i(\omega_3+\omega_2-\omega_6)-(E_{n_x}-\mu)$ by $-(E_{n_x}-\mu)=-\hbar\omega_c(n_x-N)$. The full expression gives an effective action S_{eff} in Eq. (3) with a frequency-dependent u_3, i.e., an interaction which is retarded on time scales shorter than $1/(E_{n_x}-\mu)\propto 1/\hbar\omega_c$. So long as all energies are small compared to $(E_{n_x}-\mu)$, we can neglect these retardation effects (which are higher order in κ) by making this approximation. This enables us to pass from action (4) to a Hamiltonian.

The effective two-body interaction is also renormalized at the same order so this contribution must also be kept for the sake of consistency (and may be equally important for determining the ground state). There are three second-order diagrams contributing to the renormalization of the two-body interaction, which are familiar from Fermi-liquid theory [where Shankar has dubbed them ZS, ZS′, and BCS (Ref. 25)]. They are depicted in Fig. 1(d). They give a contribution (one frequency integral has been performed in each term in this expression):

$$\begin{aligned}du_2(m_4,m_3,m_2,m_1)\delta^{\alpha\alpha'}\delta^{\beta\beta'}-du_2(m_3,m_4,m_2,m_1)\delta^{\alpha\beta'}\delta^{\beta\alpha'}&=\sum_{m_x,m_{x'}}\sum_{n_x,n_{x'}\neq N}\frac{\tilde{V}^{\alpha'\gamma';\gamma\alpha}_{4,x';x,1}\tilde{V}^{\gamma\beta';\beta\gamma'}_{x,3;2,x'}[\theta(\tilde{E}_{n_x})-\theta(\tilde{E}_{n_{x'}})]}{[i(\omega_1-\omega_4)+E_{n_x}-E_{n_{x'}})]}\\&-\sum_{m_x,m_{x'}}\sum_{n_x,n_{x'}\neq N}\frac{\tilde{V}^{\beta'\gamma';\gamma\alpha}_{3,x';x,1}\tilde{V}^{\gamma\alpha';\beta\gamma'}_{x,4;2,x'}[\theta(\tilde{E}_{n_x})-\theta(\tilde{E}_{n_{x'}})]}{[i(\omega_1-\omega_3)+E_{n_x}-E_{n_{x'}})]}\\&-\frac{1}{2}\sum_{m_x,m_{x'}}\sum_{n_x,n_{x'}\neq N}\frac{\tilde{V}^{\gamma\gamma';\beta\alpha}_{4,3;x,x'}\tilde{V}^{\alpha\beta;\gamma'\gamma}_{x,x';2,1}[\theta(\tilde{E}_{n_x})-\theta(-\tilde{E}_{n_{x'}})]}{[i(\omega_1+\omega_2)-(\tilde{E}_{n_x}+\tilde{E}_{n_{x'}})]},\end{aligned} \tag{7}$$

where $\tilde{E}_{n_x}\equiv E_{n_x}-\mu$.

A useful way of representing the results of evaluating Eqs. (6) and (7) is through Haldane's pseudopotentials[4] and their generalization to three-body (and higher) interactions by Simon *et al.*[23] The idea is to project the interaction onto states of fixed relative angular momentum within the Nth LL:

$$V^{(2)}_{M,S}\equiv\sum_{\{m_i\}}\langle M,M_{CM},S,S^z|m_3,\beta;m_4,\alpha\rangle\times\langle m_1,\alpha;m_2,\beta|M,M_{CM},S,S^z\rangle u_2(m_4,m_3,m_2,m_1).$$

Here, $|M,M_{CM},S,S^z\rangle$ is a two-electron eigenstate of relative angular momentum M, center-of-mass quantum number M_{CM}, total spin S and total spin z-component S^z. It is unimportant what basis we use for the center-of-mass wave functions since the interaction is translationally invariant so the matrix element is independent of M_{CM}. By spin-rotational symmetry, it is also independent S^z. For M odd, Fermi statistics requires $S=1$; this case is relevant to a fully polarized system. For M even, $S=0$; these pseudopotentials only play a role when both spin species are present.

Generalizing the three-body pseudopotentials of Ref. 23 slightly to include spin, we have

$$V^{(3)}_{M,S,q,q'} \equiv \sum_{\{m_i\}} \langle M,M_{CM},S,S^z,q'|m_4,\gamma;m_5,\beta;m_6,\alpha\rangle \times \langle m_1,\alpha;m_2,\beta;m_3;\gamma|M,M_{CM},S,S^z,q\rangle u_3(m_6,\ldots,m_1).$$

Here, $|M,M_{CM},S,S^z,q\rangle$, are the three-electron states of relative angular momentum M and total spin S. Unlike the two particle case, there may be more than one such state, which we label by the index q. The first few $M=3$, 5, 6, 7, 8 only have a single such state so this extra index is superfluous. For instance, $|M,M_{CM}=0,S=3/2,S^z=3/2\rangle$ for $M=3,5$ can be expressed in the $|m_1,m_2,m_3\rangle$ basis as $|3,0,3/2,3/2\rangle = |0,\uparrow;1,\uparrow;2,\uparrow\rangle$ and $|5,0,3/2,3/2\rangle=(\sqrt{3}|0,\uparrow;1,\uparrow;4\uparrow\rangle - 2|0,\uparrow;2,\uparrow;3\uparrow\rangle)/\sqrt{7}$. Similarly, $|1,0,1/2,1/2\rangle = |0,\uparrow;0,\downarrow;1,\uparrow\rangle$. From Eqs. (6) and (7), we compute the two-body and three-body pseudopotentials. Our results are displayed in Table I.

Table I contain the main results of this Rapid Communication. The most salient features are the following: (1) The effects of LL mixing are smaller than naively expected. The corrections to the pseudopotentials are proportional to κ, as expected, but the slight surprise is that the coefficient of κ is <0.2; indeed, those that don't involve reversed spins are <0.02. This small dimensionless number results from phase-space restrictions (which are important, albeit less so than in a zero-field Fermi liquid), partial cancellation between different excited LLs, and the oscillatory nature of the relevant matrix elements. Continuing our calculation to order κ^2, the contribution to $V^{(3)}_{3,3/2}$ from one-loop diagrams is $\approx 0.006\kappa^2$,[26] Thus, our calculation might be valid to larger κ than naively expected. In Ref. 10, LL mixing was considered in the Hamiltonian approach using a modified charge operator and the Zhang-Das Sarma potential. They found a 5% effect on gaps for $\kappa=1$, further evidence of the smallness of these effects. (2) The $m=3$ three-particle pseudopotential, $V^{(3)}_{3,3/2}$, is negative. This is not an *a priori* obvious result since our computation is essentially an RG calculation (*not* a calculation of the ground state energy) and the β function can be positive or negative. The physical significance of our result follows from the observation that the MR Pfaffian is the exact ground state of a Hamiltonian with $V^{(3)}_{3,3/2}>0$ and all other three-body and all two-body pseudopotentials equal to zero.[16,23] The anti-Pfaffian state is the exact ground state of the particle-hole conjugate Hamiltonian, which has $V^{(3)}_{3,3/2}<0$ and nonzero two-body pseudopotentials. As seen in a recent preprint,[27] negative $V^{(3)}_{3,3/2}$ also favors the anti-Pfaffian state over the MR Pfaffian state near the Coulomb point. The quantum phase transition between the two states is first order[27] so the effects of a small symmetry-breaking term will be magnified. Thus, the $V^{(3)}_{m,s}$s, though small, may have a large effect. However, it is possible for a different state to be lower in energy than either the anti-Pfaffian or MR Pfaffian. Thus, even though $V^{(3)}_{3,3/2}$ is larger in magnitude in the $N=0$ LL than in the $N=1$ LL (and still negative), this does not necessarily imply that the anti-Pfaffian state is expected there, too. Finally, as noted above, the order κ^2 contribution to $V^{(3)}_{3,3/2}$ is positive ($+0.006\kappa^2$) so it is possible that $V^{(3)}_{3,3/2}$ changes sign as κ is increased and a phase transition occurs.

(3) It is important to consider the higher $V^{(3)}_{m,3/2}$s, which oscillate with m. It is also important to consider the $\delta V^{(2)}_{m,1}$s, which have opposite signs in the $N=0$ and $N=1$ LLs. Landau-level mixing suppresses the ratio $V^{(2)}_{1,1}/V^{(2)}_{3,1}$ more strongly in the $N=1$ LL than in the $N=0$ LL. This may help explain why a quantum Hall plateau does not develop in the $N=0$ LL, unlike in the $N=1$ LL. (4) The pseudopotentials which come into play when the electrons in the Nth LL are not fully spin-polarized are more strongly renormalized. Thus, apart from particle-hole symmetry breaking, the effects of LL mixing may be strongest in partially polarized or unpolarized states. At any rate, our results should be viewed as an input for exact diagonalization and DMRG studies of quantum Hall states: with the effects of Landau-level mixing incorporated in the starting Hamiltonian, one could thereby solve a more realistic model.

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